

Multivariate Time Series: Part 2

Simultaneous Equations and Vector Autoregressions

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Outline

1 Multivariate Time Series

- Simultaneous Equations
- Structural Equations and Reduced Forms
- Estimation and Ordinary Least Squares
- Structural Equations and VARs
- Recall Example of Importance of “Ordering”
- Summary

Identification

- We have seen that statistical fit will not allow us to resolve identification issues
- Maybe Economics can help us out

Structural equations

- Suppose have simultaneous system for supply and demand

$$q_t^d = \alpha p_t + \beta y_t + \varepsilon_t^d, \alpha < 0, \beta > 0 \quad (1)$$

$$q_t^s = \gamma p_t + \delta w_t + \varepsilon_t^s, \gamma, \delta > 0$$

$$q_t = q_t^d = q_t^s$$

- ▶ p is price, q is quantity and y and w are other variables that affect demand and supply
- ▶ Suppose $E \varepsilon_t^d = E \varepsilon_t^s = 0$, $E \varepsilon_t^d \varepsilon_t^s = 0$, constant variances and both serially uncorrelated)
- ▶ For convenience, suppose that all variables have expected values of zero
 - ★ Deviations from means is an obvious reason
- ▶ Equations jointly determine quantities bought and sold as well as price

Structural equations

- Simultaneous system for supply and demand

$$q_t^d = \alpha p_t + \beta y_t + \varepsilon_t^d, \alpha < 0, \beta > 0$$

$$q_t^s = \gamma p_t + \delta w_t + \varepsilon_t^s, \gamma, \delta > 0$$

$$q_t = q_t^d = q_t^s$$

- Why are they called structural equations?
- Called **structural equations** because they summarize the behavior of economic agents
 - ▶ Demand equation summarizes behavior of buyers and is invariant to changes in supply
 - ▶ Supply equation summarizes behavior of sellers and is invariant to changes in demand

Structural equations

- Have simultaneous system for supply and demand

$$q_t = \alpha p_t + \beta y_t + \varepsilon_t^d, \alpha < 0, \beta > 0$$

$$q_t = \gamma p_t + \delta w_t + \varepsilon_t^s, \gamma, \delta > 0$$

- An equation is **structural** relative to an intervention if it is invariant to that intervention
 - ▶ Example: Suppose a sales tax is imposed on purchases and the demand equation is changed to include $(1 + \tau) p$ where τ is the sales tax
 - ▶ Then these equations are structural relative to that intervention

Structural equations

- Simultaneous system for supply and demand

$$q_t = \alpha p_t + \beta y_t + \varepsilon_t^d, \alpha < 0, \beta > 0$$

$$q_t = \gamma p_t + \delta w_t + \varepsilon_t^s, \gamma, \delta > 0$$

- The variables p and q are **jointly dependent**
- The variables y and w are **exogenous**
 - ▶ In the context of theories, this means a variable is “determined outside the system being examined”
 - ▶ In econometric contexts, it can mean a variable is “uncorrelated with the error term” in this and possibly other equations

Structural equations and reduced form

- Simultaneous system for supply and demand

$$q_t = \alpha p_t + \beta y_t + \varepsilon_t^d, \quad \alpha < 0, \beta > 0$$

$$q_t = \gamma p_t + \delta w_t + \varepsilon_t^s, \quad \gamma, \delta > 0$$

- Can solve for **reduced form** (with time subscript suppressed)

$$q = (\gamma - \alpha)^{-1} \left[\gamma \beta y - \alpha \delta w + \gamma \varepsilon_t^d - \alpha \varepsilon_t^s \right]$$

$$p = (\gamma - \alpha)^{-1} \left[\beta y - \delta w + \varepsilon_t^d - \varepsilon_t^s \right]$$

- ▶ Cramer's rule is the simplest way to do it, at least for me

Reduced form

- Reduced form is

$$q = (\gamma - \alpha)^{-1} \left[\gamma\beta y - \alpha\delta w + \gamma\varepsilon_t^d - \alpha\varepsilon_t^s \right]$$

$$p = (\gamma - \alpha)^{-1} \left[\beta y - \delta w + \varepsilon_t^d - \varepsilon_t^s \right]$$

- This also can be written

$$q = \pi_{11}y + \pi_{12}w + u_1 \tag{2}$$

$$p = \pi_{21}y + \pi_{22}w + u_2$$

where

$$\pi_{11} = (\gamma - \alpha)^{-1} \gamma\beta$$

$$\pi_{12} = -(\gamma - \alpha)^{-1} \alpha\delta$$

$$\pi_{21} = (\gamma - \alpha)^{-1} \beta$$

$$\pi_{22} = -(\gamma - \alpha)^{-1} \delta$$

Reduced form

- In this example with

$$\pi_{11} = (\gamma - \alpha)^{-1} \gamma \beta$$

$$\pi_{12} = -(\gamma - \alpha)^{-1} \alpha \delta$$

$$\pi_{21} = (\gamma - \alpha)^{-1} \beta$$

$$\pi_{22} = -(\gamma - \alpha)^{-1} \delta$$

- It is easy to show that given π_{ij} , we can solve for

$$\alpha = \pi_{12} / \pi_{22}$$

$$\gamma = \pi_{11} / \pi_{21}$$

$$\beta = (\gamma - \alpha) \pi_{21}$$

$$\delta = (\gamma - \alpha) \pi_{22}$$

if π_{21} and π_{22} are not equal to zero

Identification and estimation

- In this example with

$$q_t = \alpha p_t + \beta y_t + \varepsilon_t^d, \quad \alpha < 0, \beta > 0$$

$$q_t = \gamma p_t + \delta w_t + \varepsilon_t^s, \quad \gamma, \delta > 0$$

- We can estimate the parameters by estimating the reduced form by OLS and then inferring the structural parameters
- This is called **indirect least squares**
 - ▶ Not generally available but sometimes is
 - ▶ With simple equations here with serially uncorrelated errors and no lagged variables, can estimate reduced form by ordinary least squares
 - ▶ Given the estimated reduced form, solve for the structural coefficients

Indirect Least Squares

- How Indirect Least Squares works
 - ▶ Estimate reduced form by OLS and get estimates

$$\begin{array}{cc} \hat{\pi}_{11} & \hat{\pi}_{12} \\ \hat{\pi}_{21} & \hat{\pi}_{22} \end{array}$$

- ▶ These coefficients are estimated consistently because the error terms are uncorrelated with the right-hand-side variables
- Furthermore if π_{21} and π_{22} are not equal to zero

$$\hat{\alpha} = \hat{\pi}_{12} / \hat{\pi}_{22}$$

$$\hat{\gamma} = \hat{\pi}_{11} / \hat{\pi}_{21}$$

$$\hat{\beta} = (\hat{\gamma} - \hat{\alpha}) \hat{\pi}_{21}$$

$$\hat{\delta} = (\hat{\gamma} - \hat{\alpha}) \hat{\pi}_{22}$$

- ▶ and these are consistent estimators of α , γ , β , and δ
 - ★ Recall that $\text{plim } ab = \text{plim } a \text{plim } b$

OLS and estimation of structural equations

- Why can't we just estimate the structural equations by OLS?
- Suppose estimate this demand equation by OLS
 - ▶ Simplify algebra by supposing $\beta = 0$ so we have

$$q_t = \alpha p_t + \varepsilon_t^d, \alpha < 0$$

- ▶ Estimate α by

$$\hat{\alpha} = \frac{\sum q p}{\sum p^2}$$

- ▶ What is the equation for the quantity q ? It is

$$q_t = \alpha p_t + \varepsilon_t^d$$

OLS and estimation of structural equations

- Substitute this equation for q into the OLS estimating equation to get

$$\begin{aligned}\hat{\alpha} &= \frac{\sum (\alpha p_t + \varepsilon_t^d) p}{\sum p^2} \\ &= \alpha \frac{\sum p^2}{\sum p^2} + \frac{\sum \varepsilon_t^d p}{\sum p^2} \\ &= \alpha + \frac{\sum \varepsilon_t^d p}{\sum p^2}\end{aligned}$$

- Question of consistency comes down to whether

$$\text{plim} \frac{\sum \varepsilon_t^d p / T}{\sum p^2 / T} = 0?$$

OLS and estimation of structural equations

- Question of consistency comes down to whether

$$\text{plim} \frac{\sum \varepsilon_t^d p / T}{\sum p^2 / T} = \frac{\text{plim} \sum \varepsilon_t^d p / T}{\text{plim} \sum p^2 / T} = 0?$$

- But it is not. The price is

$$p = (\gamma - \alpha)^{-1} \left[-\delta w + \varepsilon_t^d - \varepsilon_t^s \right]$$

- And

$$\text{plim} \sum \varepsilon_t^d (\gamma - \alpha)^{-1} \left[-\delta w + \varepsilon_t^d - \varepsilon_t^s \right] / T \neq 0$$

in general because ε_t^d affects both the price and the quantity demanded

- This is a system of equations that simultaneously determine price and quantity

Structural estimation methods

- Structural estimation methods come down to finding estimators that either directly consider this correlation or else circumvent it in a way similar to instrumental variables

Structural equations and VARs

- Suppose have dynamic simultaneous system for supply and demand

$$q_t = [\alpha_0 + \alpha(L)] p_t + \alpha^*(L) q_t + \beta y_t + \varepsilon_t^d$$
$$q_t = [\gamma_0 + \gamma(L)] p_t + \gamma^*(L) q_t + \delta w_t + \varepsilon_t^s$$

- where

$$\alpha_0 + \alpha(L) = \alpha_0 + \sum_{i=1}^k \alpha_i L^i$$
$$\alpha^*(L) = \sum_{i=1}^k \alpha^{*i} L^i$$
$$\gamma_0 + \gamma(L) = \gamma_0 + \sum_{i=1}^k \gamma_i L^i$$
$$\gamma^*(L) = \sum_{i=1}^k \gamma^{*i} L^i$$

- Now solve for reduced form
- The lagged values are similar to exogenous variables, i.e. not solved out
- If the lagged values are uncorrelated with the error terms, they can be called **predetermined variables**

Structural equations and VARs

- Solve for reduced form

$$q_t = \Pi_{11}(L) q_{t-1} + \Pi_{12}(L) p_{t-1} + \pi_{11} y_t + \pi_{12} w_t + u_{t1}$$

$$p_t = \Pi_{21}(L) q_{t-1} + \Pi_{22}(L) p_{t-1} + \pi_{21} y_t + \pi_{22} w_t + u_{t2}$$

- where $\Pi_{11}(L) = \sum_{i=1}^k \Pi_{11,i} L^{i-1}$, and so forth
- Similar to a vector autoregression except have variables y and w
- This is a vector autoregression VAR if have one of the following
 - ▶ y and w have coefficients of zero
 - ▶ y and w are serially uncorrelated and we sweep them into the error terms
 - ▶ we expand this into a 4-variable VAR
 - ★ Maybe with just simple autoregressions for y and w if there is no reason to think that y and w depend on past prices and quantities

Structural equations and VARs

- Two-variable VAR

$$q_t = \Pi_{11} (L) q_{t-1} + \Pi_{12} (L) p_{t-1} + u_{t1}$$

$$p_t = \Pi_{21} (L) q_{t-1} + \Pi_{22} (L) p_{t-1} + u_{t2}$$

- Possible four-variable set of equations

$$q_t = \Pi_{11} (L) q_{t-1} + \Pi_{12} (L) p_{t-1} + \pi_{11} y_t + \pi_{12} w_t + u_{t1}$$

$$p_t = \Pi_{21} (L) q_{t-1} + \Pi_{22} (L) p_{t-1} + \pi_{21} y_t + \pi_{22} w_t + u_{t2}$$

$$y_t = \Pi_{33} (L) y_{t-1} + u_{t3}$$

$$w_t = \Pi_{34} (L) w_{t-1} + u_{t4}$$

- Possible four-variable VAR

$$q_t = \Pi_{11} (L) q_{t-1} + \Pi_{12} (L) p_{t-1} + \Pi_{31} (L) y_{t-1} + \Pi_{41} (L) w_{t-1} + u_{t1}$$

$$p_t = \Pi_{21} (L) q_{t-1} + \Pi_{22} (L) p_{t-1} + \Pi_{32} (L) y_{t-1} + \Pi_{42} (L) w_{t-1} + u_{t2}$$

$$y_t = \Pi_{33} (L) y_{t-1} + u_{t3}$$

$$w_t = \Pi_{34} (L) w_{t-1} + u_{t4}$$

Structural equations and VARs

- In reduced form

$$q_t = \Pi_{11}(L) q_{t-1} + \Pi_{12}(L) p_{t-1} + \Pi_{31}(L) y_{t-1} + \Pi_{41}(L) w_{t-1} + u_{t1}$$

$$p_t = \Pi_{21}(L) q_{t-1} + \Pi_{22}(L) p_{t-1} + \Pi_{32}(L) y_{t-1} + \Pi_{42}(L) w_{t-1} + u_{t2}$$

$$y_t = \Pi_{33}(L) y_{t-1} + u_{t3}$$

$$w_t = \Pi_{34}(L) w_{t-1} + u_{t4}$$

- The error terms u_1 and u_2 in the equations for p and q in the VAR are sums of the innovations in both demand and supply
 - ▶ Do not represent demand or supply shocks (innovations) alone
 - ▶ Both error terms include both demand and supply shocks

Structural equations and VARs

- Bottom line:
- A VAR can be interpreted as a reduced form of structural equations

Structural equations and recursive VARs

- A recursive system is one in which there the errors are uncorrelated and there is an ordering of variables in equations
 - ▶ As below, q_t in p_t equation but not p_t in q_t equation
- Unless the structural equations themselves are *recursive*, the VAR reduced form generally will **not** look like

$$\begin{aligned} p_t &= \pi_{10}q_t + \Pi_{21}(L)q_{t-1} + \Pi_{22}(L)p_{t-1} + \pi_{21}y_t + \pi_{22}w_t + u_{2t} \\ q_t &= \Pi_{11}(L)q_{t-1} + \Pi_{12}(L)p_{t-1} + \pi_{11}y_t + \pi_{12}w_t + u_{1t} \end{aligned}$$

or its inverse, either with error terms uncorrelated with all right-hand-side variables

- Estimating such an equation estimates neither the reduced form nor the structural equations
- Does not estimate demand equation or supply equation
- Errors are not demand shocks or supply shocks
- Errors are uncorrelated
 - ▶ It is a representation of the relationship between p_t and q_t , just not demand equation, supply equation or reduced form

Government expenditures and real GDP

- Does this matter?
- Government expenditures and real GDP from Andrew Young “Why in the world are we all Keynesians now?”, page 12
 - ▶ Graphs show response of real GDP to a one-standard deviation “shock” to government spending
 - ▶ Top graph has current government expenditures appearing in the real GDP equation

$$\begin{aligned}y_t &= a_1^* g_t + & a_{1,1} y_{t-1} + a_{1,2} g_{t-1} + \varepsilon_{1,t} \\g_t &= & a_{2,1} y_{t-1} + a_{2,2} g_{t-1} + \varepsilon_{2,t}\end{aligned}$$

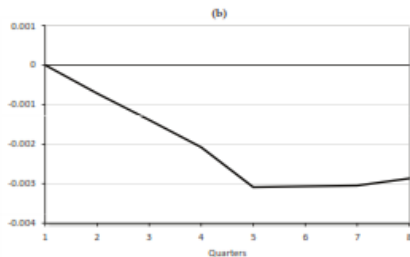
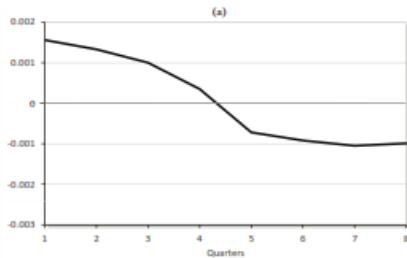
- ★ Effect of one-standard-deviation change of $\varepsilon_{2,t}$ on y_t
- ▶ Bottom graph has current GDP appearing in the current government expenditures equation

$$\begin{aligned}y_t &= & a_{1,1} y_{t-1} + a_{1,2} g_{t-1} + \eta_{1,t} \\g_t &= a_2^* y_t + & a_{2,1} y_{t-1} + a_{2,2} g_{t-1} + \eta_{2,t}\end{aligned}$$

- ★ Effect of one-standard-deviation change of $\eta_{2,t}$ on y_t

Government expenditures and real GDP

Figure 4
Estimated Effects of Real GDP to a Federal Government Expenditures Shock



Why not estimate simple VAR?

- Why not use

$$y_t = a_{1,1}y_{t-1} + a_{1,2}g_{t-1} + u_{1,t}$$

$$g_t = a_{2,1}y_{t-1} + a_{2,2}g_{t-1} + u_{2,t}$$

- Effect of one-standard-deviation change of $u_{2,t}$ on $y_{1,t}$ doesn't represent effect of change in government's desired spending alone because the error term generally is a function of all structural errors

Summary

- Economics goes a long way in identification problems
- If you don't have an economic theory for a system you are estimating, you are likely to be in trouble if go beyond forecasting
- Simultaneous equations systems are very useful when variables are simultaneously determined
- Vector autoregressions can be interpreted as reduced forms of simultaneous equation systems
- Recursive systems of variables can be poor – potentially uninformative or even misleading – representations of the economic relationships among variables
- Recursive systems of variables can be good – potentially informative with little contentious economic theory – summaries of the statistical relationships among variables
 - ▶ Have to be careful about interpretation