Why Are Vector Autoregressions Useful in Finance?

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Overview of Vector Autoregressions

- Vector autoregressions are
  - A concise way of summarizing data
  - Generally have little serial correlation in residuals
  - Can be used to examine complex relationships among variables
- Examples
  - Information content in prices in different asset markets
  - Effects of arbitrage across markets
  - Relationships between options and cash markets

Truth in Advertising

- What a Vector Autoregression (VAR) cannot do

  - It generally is not an intelligent causal model of the data

    - A VAR can summarize the data

    - Generally is not the set of equations implied by any theory

Vector Autoregression Illustrated

- A simple vector autoregression for two variables *y* and *z* is

$$y_t = \alpha^y + \beta^y y_{t-1} + \gamma^y z_{t-1} + \varepsilon_t^y$$
$$z_t = \alpha^z + \beta^z y_{t-1} + \gamma^z z_{t-1} + \varepsilon_t^z$$

  - where

    - the α's and β's are parameters

    - the epsilons are white noise, i.,e.,

    -

$$\mathrm{E}\,\varepsilon_t^i = 0, \mathrm{Var}\,\varepsilon_t^i = \sigma^2 \text{ and } \mathrm{Cov}(\varepsilon_t^i, \varepsilon_s^j) = 0$$
$$i, j = y, z \text{ and } i \neq j, t \neq s$$

Vector Autoregression
Cleaner Representation

- Matrix notation

$$\begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \alpha^y \\ \alpha^z \end{bmatrix} + \begin{bmatrix} \beta^y & \gamma^y \\ \beta^z & \gamma^z \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_t^y \\ \varepsilon_t^z \end{bmatrix}$$

  - which can be written

$$\mathbf{x_t} = \alpha + \beta \mathbf{x_{t-1}} + \varepsilon_{\mathbf{t}}$$

  - which is a vector autoregression
    - uses vectors
    - is an autoregression in the vectors

Vector Autoregression
Generalizes Easily

- More variables
  - Increase the number of rows and columns
  - Illustration
    - $\mathbf{x_t} = \boldsymbol{\alpha} + \boldsymbol{\beta}\mathbf{x_{t-1}} + \boldsymbol{\varepsilon}_\mathbf{t}$
      - Vectors (2 x 1)
      - Matrix $\boldsymbol{\beta}$ (2 x 2)
  - Could just as well be
    - $\mathbf{x_t}$, $\mathbf{x}_{\mathbf{t}-1}$, $\boldsymbol{\alpha}$, and $\boldsymbol{\varepsilon}_\mathbf{t}$
      - (3 x 1), (4 x 1), ... (n x 1)
    - $\boldsymbol{\beta}$
      - (3 x 3), (4 x 4), ... (n x n)

Vector Autoregression

Generalizes Easily

- More lags of variables

$$y_t = \alpha^y + \beta_1^y y_{t-1} + \beta_2^y y_{t-2}$$
$$+ \gamma_1^y z_{t-1} + \gamma_2^y z_{t-2} + \varepsilon_t^y$$
$$z_t = \alpha^z + \beta_1^z y_{t-1} + \beta_2^z y_{t-2}$$
$$+ \gamma_1^z z_{t-1} + \gamma_2^z z_{t-2} + \varepsilon_t^y$$

- Two ways
  - Polynomials in lag operator
  - Augmented matrices

Augmented Matrices

- Original equations

$$y_t = \alpha^y + \beta_1^y y_{t-1} + \beta_2^y y_{t-2}$$
$$+ \gamma_1^y z_{t-1} + \gamma_2^y z_{t-2} + \varepsilon_t^y$$
$$z_t = \alpha^z + \beta_1^z y_{t-1} + \beta_2^z y_{t-2}$$
$$+ \gamma_1^z z_{t-1} + \gamma_2^z z_{t-2} + \varepsilon_t^y$$

  - Want

$$\mathbf{x_t} = \boldsymbol{\alpha} + \boldsymbol{\beta} \mathbf{x_{t-1}} + \boldsymbol{\varepsilon}_\mathbf{t}$$

  - Add variables in $\mathbf{x_t}$ so that have $\mathbf{x_t}$ and $\mathbf{x_{t-1}}$ progress naturally in time
    - $\mathbf{x_{t-1}}$ will become $\mathbf{x_t}$ next period

Augmented Matrices Construction

- Add variables to $\mathbf{x_t}$ and $\mathbf{x_{t-1}}$

  Augment $\mathbf{x_t}$ and $\mathbf{x_{t-1}}$

  - $\mathbf{x_{t-1}}$ has to include $y_{t-1}, y_{t-2}, z_{t-1}, z_{t-2}$

    - So include them

    - $\mathbf{x_{t-1}}' = [y_{t-1}, y_{t-2}, z_{t-1}, z_{t-2}]$

  - Implies that

    - $\mathbf{x_t}' = [y_t, y_{t-1}, z_t, z_{t-1}]$

  - What is vector $\boldsymbol{\alpha}$?
  - What is matrix $\boldsymbol{\beta}$?

Contents of Augmented Matrices

- Just fill in the blanks in **α** and **β**

$$\begin{bmatrix} y_t \\ y_{t-1} \\ z_t \\ z_{t-1} \end{bmatrix} = \begin{bmatrix} \alpha^y \\ 0 \\ \alpha^z \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \beta_1^y & \beta_2^y & \gamma_1^y & \gamma_2^y \\ ? & ? & ? & ? \\ \beta_1^z & \beta_2^z & \gamma_1^z & \gamma_1^z \\ ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ z_{t-1} \\ z_{t-2} \end{bmatrix} + \begin{bmatrix} \varepsilon_t^y \\ 0 \\ \varepsilon_t^z \\ 0 \end{bmatrix}$$

Contents of Augmented Matrices Completed

$$\begin{bmatrix} y_t \\ y_{t-1} \\ z_t \\ z_{t-1} \end{bmatrix} = \begin{bmatrix} \alpha^y \\ 0 \\ \alpha^z \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \beta_1^y & \beta_2^y & \gamma_1^y & \gamma_2^y \\ - & - & - & - \\ \beta_1^z & \beta_2^z & \gamma_1^z & \gamma_1^z \\ - & - & - & - \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ z_{t-1} \\ z_{t-2} \end{bmatrix} + \begin{bmatrix} \varepsilon_t^y \\ 0 \\ \varepsilon_t^z \\ 0 \end{bmatrix}$$

- Could get rid of $\boldsymbol{\alpha}$ by just including constant in variables in $\mathbf{x}$
  - We'll just suppress constants
  - Deviations from means would let us ignore constants

Properties of First-order Representation

- Matrix $\boldsymbol{\beta}$ is not singular
  - despite the augmentation

- If the ε's are serially uncorrelated, then ordinary least squares is a consistent estimator of

  - the parameters in $\boldsymbol{\beta}$

- If the ε's are serially uncorrelated and homoskedastistic, then ordinary least squares is a consistent estimator of

  - the variance of the ε's

  - the serial correlation of the ε's

- Can say these things without worrying about "stationarity" and "unit roots"

Unit Roots

- What are the implications of unit roots?

- Unibiasedness
-
  - Proving unbiasedness not an issue

$$\mathrm{E}\,\beta^{ols}_{x_t, x_{t-1}} = \mathrm{E}\frac{\sum x_t x_{t-1}}{\sum x_{t-1}^2}$$

$$\neq \frac{\mathrm{E}\sum x_t x_{t-1}}{\mathrm{E}\sum x_{t-1}^2}$$

  - whereas

$$\operatorname{plim}\beta^{ols}_{x_t, x_{t-1}} = \operatorname{plim}\frac{\sum x_t x_{t-1}}{\sum x_{t-1}^2}$$

$$= \frac{\operatorname{plim}\sum x_t x_{t-1}}{\operatorname{plim}\sum x_{t-1}^2}$$

Unit Roots

- Suppose no unit roots
  - Additional assumptions
    - Homoskedastic errors
    - No serial correlation of errors
  - Ordinary least squares estimator of $\boldsymbol{\beta}$
  - Consistent
  - OLS estimates of standard errors are correct in the sense that
    - Estimated coefficients divided by their standard deviations are asymptotically normal under the null hypothesis that the coefficients are zero
    - Standard F-statistics have an F distribution

Unit Roots

- Suppose that variables have unit roots

    - e.g. $x_t = x_{t-1} + \varepsilon_t^x$

    - Then coefficient of at least one variable relative to its standard deviation will not have a normal distribution

- Solution depends on issue of cointegration

Cointegration

- Definition of cointegrated variables
  - Suppose that *y* and *z* have unit roots
  - *y* and *z* are cointegrated if, e.g., $y_t - \delta z_t$ does not have a unit root ($\delta \neq 0$)

Estimation Depending on Cointegration

- Suppose that variables **not cointegrated**
  - Estimate VAR in the first differences
  - Δx_t and Δy_t wherever x and y appear above

- Suppose that variables are **cointegrated**
  - Estimate Vector Error Correction Mechanism (VECM)
    - VAR with additional terms for cointegrating vectors

Vector Error Correction Mechanism

- VECM with two variables and two lags is

$$\Delta y_t = \lambda_y (y_{t-1} - \delta z_{t-1}) + \beta_1^y \Delta y_{t-1} + \beta_2^y \Delta y_{t-2}$$

$$+ \gamma_1^y \Delta z_{t-1} + \gamma_2^y \Delta z_{t-2} + \varepsilon_t^y$$

$$\Delta z_t = \lambda_y (y_{t-1} - \delta z_{t-1}) + \beta_1^z \Delta y_{t-1} + \beta_2^z \Delta y_{t-2}$$

$$+ \gamma_1^z \Delta z_{t-1} + \gamma_2^z \Delta z_{t-2} + \varepsilon_t^y$$

  - Same as VAR with a term added that is the cointegrating vector

  - Why one cointegrating vector?
    Why not two?
    - If N variables all have unit roots, then there can be at most N-1 cointegrating vectors
    - Hence, one cointegrating vector here

Summary

- Many other details

- Basic idea is that a VAR or VECM summarizes the correlations in the data

- VAR or VECM is especially useful when variables are serially correlated

Applications

- Can use VAR or VECM to obtain estimate of the response of one variable when another changes
  - This is a tricky issue .
  - Simplest if correlation of errors is zero across equations in VAR or VECM
- Can use VAR or VECM to examine whether one variable helps to predict the other
  - “Granger causality”
  - Can answer substantive questions such as whether price on one of two markets reflects information faster

Applications

- Dividends and stock prices

  - Let *d* be the logarithm of real dividends and *p* be the logarithm of real stock prices

  - Suppose that dividends and stock prices are cointegrated with a coefficient of one - $d - p$ does not have a unit root

  - VECM for *d* and *p*

$$\Delta d_t = \lambda_y (d_{t-1} - p_{t-1}) + \beta_1^y \Delta d_{t-1} + \beta_2^y \Delta d_{t-2}$$

$$+ \gamma_1^y \Delta p_{t-1} + \gamma_2^y \Delta p_{t-2} + \varepsilon_t^y$$

$$\Delta p_t = \lambda_y (d_{t-1} - p_{t-1}) + \beta_1^z \Delta d_{t-1} + \beta_2^z \Delta d_{t-2}$$

$$+ \gamma_1^z \Delta p_{t-1} + \gamma_2^z \Delta p_{t-2} + \varepsilon_t^y$$

Applications (continued)

- Price of cash and future
  - For example, stock price index and futures price
    - Want high-frequency data for this
    - Probably are cointegrated with a coefficient of one
    - Similar VECM to one for dividends and stock prices
    - When done, may want more elaborate model that allows for on a nonlinear relationship

Applications (continued)

- Stock price index and volatility of stock index from options market
  - No obvious reason why they should be cointegrated
  - VAR would be more appropriate if not cointegrated
  - It might pay to look at the relationship between the implied volatility more carefully
    - Context of option-pricing models
    - Nonlinear models

Conclusion

- Vector autoregressions and Vector error correction mechanisms can be very useful

- They can be useful summaries of the relationship between variables

- They are not a substitute for thought

References

There are many suitable references for the level of generality of this lecture. One such reference is

Greene, William H. 2000. *Econometric Analysis*. Fourth edition. New York: Prentice Hall, Chapter 17.

Another reference that has more detail and more English is

Kennedy, Peter. 1998. *A Guide to Econometrics*. Fourth edition. Cambridge: The MIT Press, Chapter 17.