Monetary Economics

Dynamic Programming

Gerald P. Dwyer

September 2021

Outline

Deterministic Infinite-Horizon Setting

- Economy runs for $t = 0, ..., \infty$
- Only households
  - Sometimes characterized as a "Robinson Crusoe economy"
- Perfect foresight

Deterministic Infinite-Horizon Problem

- The objective each period is of the form

$$u(c_t)$$

with utility a function of current consumption with utility function an increasing, concave function

- The overall objective is to maximize

$$U_t = \sum_{i=0}^{\infty} \beta^i u(c_{t+i})$$

with β a discount factor between zero and one

- Can solve this problem as we did for Ricardian Equivalence
  - Maximize this function with respect to c_{t+i} and k_{t+1+i}, subject to a budget constraint with a production function $y_{t+i} = f(k_{t+i})$

$$c_{t+i} + k_{t+1+i} - k_{t+i} = f(k_{t+i})$$

where y_{t+i} is real income each period

A Simpler Way to Solve This Particular Problem

- The objective is to maximize

$$U_t = \sum_{i=0}^{\infty} \beta^i u(c_{t+i})$$

  with β a discount factor between zero and one

- Want to maximize this function subject to a budget constraint with a production function $y_{t+i} = f(k_{t+i})$

$$c_{t+i} + k_{t+1+i} - k_{t+i} = f(k_{t+i})$$

  where y_{t+i} is real income each period

Optimum in Deterministic Infinite-Horizon Problem

- The objective is

$$\max_{c_{t+i}} U_t = \sum_{i=0}^{\infty} \beta^i u\left(c_{t+i}\right)$$

subject to

$$c_{t+i} + k_{t+1+i} - k_{t+i} = f\left(k_{t+i}\right)$$

- Substitute budget constraint into utility function and get

$$\max_{k_{t+1+i}} U_t = \sum_{i=0}^{\infty} \beta^i u\left[f\left(k_{t+i}\right) - k_{t+1+i} + k_{t+i}\right]$$

where $\max_{k_{t+1+i}}$ means maximize with respect to the sequence of capital stocks k_{t+1+i} for $i = 0, \infty$

Optimum in Deterministic Infinite-Horizon Problem

- Substitute budget constraint into utility function and get

$$\max_{k_{t+1+i}} U_t = \sum_{i=0}^{\infty} \beta^i u \left[f\left(k_{t+i}\right) - k_{t+1+i} + k_{t+i} \right]$$

- First-order conditions are first partial derivatives with respect to k_{t+1+i}
  - Capital stock k_t is at beginning of period t and determined by choices in $t-1$
  - Choose k_{t+1} in period t
  - k_t is an example of a state variable – a variable that is predetermined at the start of the period
  - k_{t+1} (as are k_{t+1+i} for all i) is an example of a control variable – a variable that is chosen to maximize an objective function

First-order Conditions in Deterministic Infinite-Horizon Problem

- Solve

$$\max_{k_{t+1+i}} U_t = \sum_{i=0}^{\infty} \beta^i u\left(f\left(k_{t+i}\right) - k_{t+1+i} + k_{t+i}\right)$$

- First-order conditions

$$-u'\left[f\left(k_{t+i}\right) - k_{t+1+i} + k_{t+i}\right] \\ +\beta u'\left[f\left(k_{t+1+i}\right) - k_{t+2+i} + k_{t+1+i}\right]\left[f'\left(k_{t+1+i}\right) + 1\right] = 0$$

First-order Conditions in Deterministic Infinite-Horizon Problem

- Now simplify

$$
\begin{aligned}
-u' \left[f\left(k_{t+i}\right) - k_{t+1+i} + k_{t+i} \right] & \\
+\beta u' \left[f\left(k_{t+1+i}\right) - k_{t+2+i} + k_{t+1+i} \right] \left[f'\left(k_{t+1+i}\right) + 1 \right] &= 0 \\
-u'\left(c_{t+i}\right) + \beta u'\left(c_{t+1+i}\right) \left[f'\left(k_{t+1+i}\right) + 1 \right] &= 0 \\
\beta = \frac{u'\left(c_{t+i}\right)}{u'\left(c_{t+1+i}\right)} \frac{1}{1 + f'\left(k_{t+1+i}\right)} &
\end{aligned}
$$

- Stationary state has $c_{t+i} = c_{t+1+i} = c$ and $k_{t+i} = k_{t+1+i} = k$ and therefore $u'\left(c_{t+i}\right) = u'\left(c_{t+1+i}\right) = u'(c)$
- which implies that

$$\beta = \frac{1}{1 + f'(k)}$$

- and with $\beta = \frac{1}{1+\delta}$

$$\delta = f'(k)$$

Value function

- The value function is a simpler way to solve this problem

Value Function

- Given the optimal value of k_{t+1+i}, $i = 0., ..., \infty$, we can write

$$V(k_t) = \max_{k_{t+1+i}} \sum_{i=0}^{\infty} \beta^i u\left[f(k_{t+i}) - k_{t+1+i} + k_{t+i}\right]$$

where $\max_{k_{t+1+i}}$ indicates maximization over the sequence $\{k_{t+1+i}\}$ for $i = 0, ..., \infty$

- Note that we also can write, one period ahead

$$V(k_{t+1}) = \max_{k_{t+2+i}} \sum_{i=0}^{\infty} \beta^i u\left[f(k_{t+1+i}) - k_{t+2+i} + k_{t+1+i}\right]$$

Recursive Structure of the Utility Function

- Given the intertemporal utility function, it always is true that

$$
\begin{aligned}
U_t &= \sum_{i=0}^{\infty} \beta^i u\left[c_{t+i}\right] \\
&= u\left(c_t\right) + \sum_{i=1}^{\infty} \beta^i u\left[c_{t+i}\right] \\
&= u\left(c_t\right) + \beta \sum_{i=0}^{\infty} \beta^i u\left[c_{t+1+i}\right] \\
&= u\left(c_t\right) + \beta U_{t+1}
\end{aligned}
$$

- The recursive structure of preferences and the economy help to simplify the problem a lot
- A difference equation in utility

Recursive Structure of the Value Function

- By definition

$$V\left(k_t\right) = \max_{k_{t+1+i}} \sum_{i=0}^{\infty} \beta^i u\left[f\left(k_{t+i}\right) - k_{t+1+i} + k_{t+i}\right]$$

- and also

$$V\left(k_{t+1}\right) = \max_{k_{t+2+i}} \sum_{i=0}^{\infty} \beta^i u\left[f\left(k_{t+1+i}\right) - k_{t+2+i} + k_{t+1+i}\right]$$

- which implies

$$V\left(k_t\right) = \max_{k_{t+1}} \left\{u\left[f\left(k_t\right) - k_{t+1} + k_t\right] + \beta V\left(k_{t+1}\right)\right\}$$

- A difference equation in the maximized value of utility

Solving for Optimum

- We know that, at the optimum,

$$V(k_t) = \max_{k_{t+1}} \{u[f(k_t) - k_{t+1} + k_t] + \beta V(k_{t+1})\}$$

- If we know $\beta V(k_{t+1})$, then it is not so obviously difficult to compute $V(k_t)$
- Just a matter of maximizing with respect to k_{t+1}
  - We suppose that we will follow the optimal path in $t+1$ and on
  - Can't just suppose it though, have to know it and how it is affected by k_{t+1} to say more than just writing down an equation
  - k_{t+1} is a state variable from the standpoint of period $t+1$ but not in t

Solving for Optimum

- We know that, at the optimum,

$$V(k_t) = \max_{k_{t+1}} \{u[f(k_t) - k_{t+1} + k_t] + \beta V(k_{t+1})\}$$

- Maximizing with respect to k_{t+1} implies that

$$V'(k_t) = -u'[f(k_t) - k_{t+1} + k_t] + \beta V'(k_{t+1}) = 0$$

- Does $\beta V'(k_{t+1})$ have properties such that this equation exists and is a maximum?
- If we know $\beta V'(k_{t+1})$, then we can solve for the equilibrium
- Solved by Benveniste and Scheinkman (1979) and more generally by Milgrom and Segal (2002)
- They also show conditions under which the function $V'(k_t)$ evaluated at k_{t+1} gives the partial derivative, conditions satisfied in typical functions used in macroeconomics

Restrictiveness of this Problem and Solution

- The problem above is simple because only one control variable (or choice variable) and only one state variable
- Also, the control variable this period is the state variable next period

More general version of problem and solution

- More generally, let

$$V(x_t) = \max_{y_s} \sum_{s=t}^{\infty} \beta^{(s-t)} F(x_s, y_s)$$

- subject to

$$x_{s+1} = G(x_s, y_s)$$

- The variables x and y can be vectors of variables
- Using a similar recursive argument, we can get

$$V(x_t) = \max_{y_t} \left[F(x_t, y_t), +\beta V(x_{t+1}) \right]$$

- subject to

$$x_{s+1} = G(x_s, y_s)$$

Eliminating the budget constraint

- Eliminating the budget constraint

$$V(x_t) = \max_{y_t} \left[F(x_t, y_t), + \beta V(G(x_t, y_t)) \right]$$

- The solution for the optimal "policy" is

$$y_t = H(x_t)$$

- and the first-order condition is

$$F_y(x_t, y_t) + \beta V' [G(x_t, y_t)] \, G_y(x_t, y_t)$$

Value function with multiple variables

- Example of value function with multiple variables in handout

Summary

- The value function provides a convenient way to solve for equilibrium in a dynamic programming model with an infinite horizon
- Set up the problem so that tomorrow looks like today other than the state variables
- Maximizing tomorrow is the same as maximizing today in terms of the structure
- This recursive setup makes it possible to solve for dynamic equilibria in some circumstances
  - Not just in steady states or stationary states
  - Also in stochastic models

Summary

- The value function provides a convenient way to solve for equilibrium in a dynamic programming model with an infinite horizon
- Set up the problem so that tomorrow looks like today other than the state variables
- Maximizing tomorrow is the same as maximizing today in terms of the structure
- This recursive setup makes it possible to solve for dynamic equilibria in some circumstances
  - Not just in steady states or stationary states
  - Also in stochastic models
- We will not pursue it but this setup is useful for finding numerical equilibrium paths for given underlying objective and constraint functions