Financial Econometrics

Factor Models of Returns

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Factor model of returns

- Simple equation for return on asset i as a function of m factors

$$r_{it} = \alpha_i + \beta_{i1} f_{1t} + ... + \beta_{im} f_{mt} + \varepsilon_{it}, \quad i = 1, ..., k, \quad t = 1, ..., T$$

  - f_{jt} is factor j in period t and there are m factors
  - The factors are common to all returns, e.g. risk factors
  - α_i is the constant term and the coefficients β_{i1} are called "factor loadings"

- One convenient matrix representation for each period is

$$\mathbf{r}_t = \boldsymbol{\alpha} + \boldsymbol{\beta} \mathbf{f}_t + \boldsymbol{\varepsilon}_t$$

where

  - $\mathbf{r}_t$ is a k by one vector of returns at t
  - $\boldsymbol{\alpha}$ is a k by one vector of constant terms
  - $\boldsymbol{\beta}$ is a k by m vector of factor loadings
  - $\mathbf{f}_t$ is a k by one vector of factors
  - $\boldsymbol{\varepsilon}_t$ is a k by one vector of error terms

Statistical properties

- In

$$r_{it} = \alpha_i + \beta_{i1} f_{1t} + \ldots + \beta_{im} f_{mt} + \varepsilon_{it}, \quad i = 1, \ldots, k, \quad t = 1, \ldots, T$$

we generally assume the factors are stationary with

$$\begin{aligned} \mathrm{E}\,\mathbf{f}_t &= \boldsymbol{\mu}_f \\ \mathrm{Cov}\left[\mathbf{f}_t\right] &= \Sigma_f \end{aligned}$$

where $\boldsymbol{\mu}_f$ is an n by 1 vector and Σ_f is an m by m matrix
  - Σ_f is not bold because it does not render correctly
- The innovations are stationary with

$$\begin{aligned} \mathrm{E}\,\varepsilon_{it} &= 0 \\ \mathrm{Cov}\left[\varepsilon_{it}, \varepsilon_{js}\right] &= \left\{ \begin{array}{c} \sigma_i^2 \text{ if } i = j \text{ and } t = s \\ 0 \text{ if } i \neq j \text{ or } t \neq s \end{array} \right\} \\ \mathrm{Cov}\left[\varepsilon_{it}, f_{js}\right] &= 0 \text{ for all } i \text{ and } j \text{ and } t \text{ and } s \end{aligned}$$

  - The ε_{it} are idiosyncratic factors
- Generally $T \gg$ number of assets k and number of factors m

Macroeconometric factor models

- The general representation of factor models is

$$r_{it} = \alpha_i + \beta_{i1} f_{1t} + ... + \beta_{im} f_{mt} + \varepsilon_{it}, \quad i = 1, ..., k, \quad t = 1, ..., T$$

- Suppose that the factors are observed macroeconomic variables such as GDP, the return on the stock market, the return on bonds, etc.
- Then this is a general "macroeconometric factor model"

Single-factor stock market models

- The general representation of factor models is

$$r_{it} = \alpha_i + \beta_{i1} f_{1t} + \ldots + \beta_{im} f_{mt} + \varepsilon_{it}, \quad i = 1, \ldots, k, \quad t = 1, \ldots, T$$

- The market model
  - Suppose that the assets are k stocks
  - Let there be one factor f_{1t}, where f_{1t} is the return on the value-weighted market portfolio
  - Then the factor model is the "market model"
- CAPM
  - Let r_{it} be the return on a stock less the risk-free rate
  - Let f_{1t} be the return on the value-weighted portfolio of stocks less the risk-free rate
  - Then the factor model is the Capital Asset Pricing Model

Multi-Factor macroeconometric factor models

- The general representation of factor models is

$$r_{it} = \alpha_i + \beta_{i1} f_{1t} + \ldots + \beta_{im} f_{mt} + \varepsilon_{it}, \quad i = 1, \ldots, k, \quad t = 1, \ldots, T$$

- The multi-factor model due to Chen, Ross and Roll
  - Let the returns be excess returns on stocks
  - Let the factors be estimated macroeconomic variables such as GDP minus expected GDP, inflation minus expected inflation and similar "unexpected" factors
  - First estimate regressions of variables that help to predict the variables
    - A VAR would be one way to do this
    - Or possibly an ECM
  - Then get estimated factors from actual values minus predicted values
  - Then estimate regressions for each of the stocks to estimate their sensitivity to the factors

Fundamtental Factor models

- The general representation of factor models is

$$r_{it} = \alpha_i + \beta_{i1} f_{1t} + ... + \beta_{im} f_{mt} + \varepsilon_{it}, \quad i = 1, ..., k, \quad t = 1, ..., T$$

- Fundamental factor models use firm-specific variables as the factors
  - Let the returns be excess returns on stocks
  - Let the factors be estimated firm-specific variables such as industry, book to market value, size and similar variables
  - Two approaches
    - Barra
    - Fama-French

Barra factor model

- The general representation of factor models is

$$r_{it} = \alpha_i + \beta_{i1} f_{1t} + ... + \beta_{im} f_{mt} + \varepsilon_{it}, \quad i = 1, ..., k, \quad t = 1, ..., T$$

- The Barra approach uses mean-adjusted returns, so the vector representation is

$$\widetilde{\mathbf{r}}_t = \boldsymbol{\beta} \mathbf{f}_t + \boldsymbol{\varepsilon}_t$$

where $\widetilde{\mathbf{r}}_t$ are mean-adjusted returns
  - Let the variables be estimated firm-specific variables such as industry, book to market value, size and similar variables
  - Treat the variables as the factor loadings and interpret them as firm-specific variables and factor loadings β_{ij}
  - Object is to estimate f_t
  - Can't estimate by OLS because the errors are heteroskedastic
    - **Estimate a cross-section regression for each period**
    - Then calculate firm-specific standard deviation of residuals across periods
    - Then use this in cross-section Generalized Least Squares Regression (GLS) for each period

Fama-French factor model

- The general representation of factor models is

$$r_{it} = \alpha_i + \beta_{i1} f_{1t} + \ldots + \beta_{im} f_{mt} + \varepsilon_{it}, \quad i = 1, \ldots, k, \quad t = 1, \ldots, T$$

- The Fama-French approach uses three factors
  - Market excess return
  - Book to market (value)
  - Size (small firms relative to large firms)
- Factor variables for regression for latter two variables are included as returns to the factor
  - Take a firm-specific variable such as the ratio of book to market
  - Sort the assets by book to market
  - Form a portfolio that is long in the top fifth of the sorted returns and short in the bottom fifth of the sorted returns
  - The observed return on this portfolio is the factor realization f_{jt}
- **Estimate a time-series regression for each firm to obtain factor sensitivities**, β_{ij}