

Financial Econometrics

Continuous Time Processes

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Why continuous time?

- Time itself is continuous
 - No matter how short the time interval, we can think of a shorter one
 - Seconds
 - Hundredths of a second
 - Nanoseconds
- Beyond this somewhat abstract argument, trading itself is not discrete separated by intervals
- Observe prices at discrete points in time
 - Can observe multiple trades in a second
 - Not necessarily at the same time interval
- We will go over the basics on estimation, but only that

- Wiener process $\{w_t\}$ where w_t is a continuous-time stochastic process
 - $\Delta w_t = \varepsilon\sqrt{\Delta t}$ where ε is a standard normal random variable and Δt is a “small” increment in time
 - Evolution of changes over time and associated randomness
 - Δw_t is independent of w_j for all $j \leq t$
 - This property says that Δw_t is independent of current and past values of w
- First condition implies that $\Delta w_t \sim N(0, \Delta t)$

Continuous-time stochastic processes as a limit

- Wiener process $\{w_t\}$ where w_t is a continuous-time stochastic process
 - $\Delta w_t = \varepsilon\sqrt{\Delta t}$ where ε is a standard normal random variable and Δt is a “small” increment in time
 - Δw_t is independent of w_j for all $j \leq t$
- Second condition can be used to derive continuous process from a limiting argument
 - Consider the evolution of w_t with $w_0 = 0$ for convenience
 - $w_t - w_0$ is the sum of many small increments; define the number of increments by $T = t/\Delta t$ so $w_t = w_{T\Delta t}$

$$w_{T\Delta t} - w_0 = \sum_{i=1}^T \Delta w_i = \sum_{i=1}^T \varepsilon_i \sqrt{\Delta t}$$

$$E w_i - E w_0 = 0$$

$$\text{Var} [w_i] = \text{Var} \left[\sum_{i=1}^T \varepsilon_i \sqrt{\Delta t} \right] = t$$

$$w_t \sim N(0, t)$$

Limiting equation for this continuous-time stochastic process

- Wiener process

$$d p_t = \sigma d w_t$$

- Generalized Wiener process for nonzero mean

$$d p_t = \mu d t + \sigma d w_t$$

- For a discrete time interval t

$$p_t - p_0 = \mu t + \sigma \varepsilon \sqrt{t}$$

$$E [p_t - p_0] = \mu t$$

$$\text{Var} [p_t - p_0] = \sigma^2 t$$

- μ is called the “drift” of the process and σ is called the “volatility” of the process
- Ito process more general with μ and σ functions of the stochastic process p_t

$$d p_t = \mu (p_t, t) d t + \sigma (p_t, t) d w_t$$

- Ito's lemma is a differentiation rule for continuous-time processes
- For the process

$$d p_t = \mu (p_t, t) d t + \sigma (p_t, t) d w_t$$

- Let $f(p_t, t)$ be some function of p_t and t
- Then

$$d f(p, t) = \frac{\partial f}{\partial p} d p + \frac{\partial f}{\partial t} d t + \frac{1}{2} \frac{\partial^2 f}{\partial p^2} (d p)^2$$

Ito's lemma applied

- For the process $dp_t = \mu dt + \sigma dw_t$
- Suppose that p_t is the log price of an asset, P_t is the price of an asset, so $P_t = \exp p_t$ and μ and σ are constants

- Ito's lemma $df(p, t) = \frac{\partial f}{\partial p} dp + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial p^2} (dp)^2$
- by way of

$$\frac{\partial P}{\partial p} = \exp p, \quad \frac{\partial P}{\partial t} = 0, \quad \frac{\partial^2 P}{\partial p^2} = \exp p, \quad (dp)^2 = \sigma^2 dt$$

- implies that the asset price evolves according to

$$\begin{aligned} dP &= e^p dp + \frac{1}{2} e^p (dp)^2 \\ &= P dp + \frac{1}{2} P (dp)^2 \\ &= P(\mu dt + \sigma dw_t) + \frac{1}{2} P (\sigma^2 dt) \\ &= \left(\mu + \frac{1}{2} \sigma^2 \right) P dt + \sigma P dw \end{aligned}$$

Ito's lemma for a log-normal stochastic process for the price

- The evolution of the return

$$\frac{dP}{P} = \left(\mu + \frac{1}{2}\sigma^2 \right) dt + \sigma dw$$

where μ is the mean log return and σ^2 is the variance of the log return

- $E \frac{dP}{P} = \mu + \frac{1}{2}\sigma^2$
- Log normal distribution is preferable for most purposes because it does not imply negative prices
- If this were all that comes out of Ito's lemma, it would not be worth messing with
- This formula for the mean is the same formula we can derive from statistics for a log-normal distribution
- For most stocks, the translation from mean log returns to mean proportional returns is not particularly important

Expected return versus mean log return

- CRSP 1984 through 2007
 - Average arithmetic return is 0.01048
 - Mean log return per month 0.00944 and variance of log return 0.00201
 - Estimate of expected return from $\frac{dP}{P} = \left(\mu + \frac{1}{2}\sigma^2\right) dt + \sigma dw$ is $0.00944 + (1/2)0.00201 = 0.01044$
- Return on personal computer companies' stocks 1984 through 2007
 - Average arithmetic return is -0.00758
 - Mean log return per month -0.00251 and variance of log return 0.02536
 - Estimate of expected return from $\frac{dP}{P} = \left(\mu + \frac{1}{2}\sigma^2\right) dt + \sigma dw$ is $-0.00251 + (1/2)0.02536 = 0.01017$

- Continuous-time stochastic processes underlie options pricing and asset pricing generally
 - Arbitrage-free pricing of alternative assets including options
 - Prices of an asset equal the prices of components that are priced elsewhere

- Problem seen in continuous-time stochastic processes relative to data
 - Too many apparent large changes in prices
 - Skewness and excess kurtosis
 - Serial correlation
- Jumps added to diffusion process
- Time-varying volatility