

Financial Econometrics

Multivariate Volatility

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Multivariate GARCH

- Multivariate return series

$$\mathbf{r}_t = \boldsymbol{\mu}_t + \mathbf{u}_t$$

where the vectors are simple generalizations of a univariate process

- The vectors are m by one with m asset returns
- $\boldsymbol{\mu}_t = E[\mathbf{r}_t | F_{t-1}]$
- \mathbf{u}_t is the innovation in the returns in period t with $E \mathbf{u}_t = \mathbf{0}$ and

$$\boldsymbol{\Sigma}_t = \text{Cov}[\mathbf{u}_t | F_{t-1}]$$

- $\boldsymbol{\Sigma}_t$ is m by m with $m(m+1)/2$ distinct elements
- The number of parameters in $\boldsymbol{\Sigma}_t$ increases with the square of m because there are $(m^2 + m) / 2$ distinct elements
 - The number of observations is mT and increases linearly with m

Curse of Dimensionality

Estimation of all parameters unconstrained not feasible for large systems

- The implication of the number of parameters in Σ_t increasing with the square of m
 - Five assets implies $m(m+1)/2 = 10$ parameters
 - Twenty assets implies $m(m+1)/2 = 210$ parameters
 - One hundred assets implies $m(m+1)/2 = 5,050$ parameters
- Suppose have 250 observations on each asset
 - Five assets implies 1,250 observations
 - Twenty assets implies 5,000 observations
 - One hundred assets implies 25,000 observations
- Even one hundred assets is not a lot
- If have 1,000 assets
 - Then 500,500 parameters in the covariance matrix
 - Only 250,000 observations
 - Probably makes more sense to look at volatility of portfolio anyway in many circumstances

- Want to find a way to allow for correlations but not have too many parameters estimated with poor estimates

Strategies to estimate multivariate volatility models

- Exponentially smoothing estimate
 - Includes all the parameters in the variance-covariance matrix
 - Adds only one additional parameter to get a time-varying variance-covariance matrix
- Multivariate GARCH models
 - Diagonal VEC model
 - GARCH(1,1) for each term in variance-covariance matrix
- Reparameterizations
 - Replace covariances by correlations
 - Cholesky decomposition
- GARCH models for bivariate returns
 - Constant correlation models
 - Time-varying correlation models

Exponential smoothing estimate

- Exponential smoothing forecasts are based on simple forecasts that smooth values of the data
 - E.g. the forecast at $t - 1$ of the value of a series x_t is
$${}_{t-1}f_t = (1 - \lambda) x_{t-1} + \lambda {}_{t-2}f_{t-1}$$
 - where ${}_{t-1}f_t$ is the forecast at $t - 1$ for period t
- Apply this idea to the variance-covariance matrix
 - Estimate time-varying variance-covariance matrix as

$$\hat{\Sigma}_t = \frac{1 - \lambda}{1 - \lambda^{t-1}} \sum_{j=1}^{t-1} \lambda^{j-1} \mathbf{u}_{t-j} \mathbf{u}'_{t-j}, \quad 0 < \lambda < 1$$

- The weights satisfy $\sum_{j=1}^{t-1} \frac{(1-\lambda)\lambda^{j-1}}{1-\lambda^{t-1}} = 1$ by construction
- As $t \rightarrow \infty$, $\hat{\Sigma}_t \rightarrow (1 - \lambda) \mathbf{u}_{t-1} \mathbf{u}'_{t-1} + \lambda \hat{\Sigma}_{t-1}$

Estimation of exponential smoothing estimate

- Can estimate parameters by maximum likelihood
 - Likelihood function assuming an underlying normal distribution is

$$\ln L(\Theta, \lambda) \propto -\frac{1}{2} \sum_{t=1}^T \ln |\Sigma_t| - \frac{1}{2} \sum_{t=1}^T (\mathbf{r}_t - \boldsymbol{\mu}_t) \Sigma_t^{-1} (\mathbf{r}_t - \boldsymbol{\mu}_t)'$$

- Don't know value of Σ_t
- Know value of $\hat{\Sigma}_t$
- Substitute $\hat{\Sigma}_t$ for Σ_t as compute sums
 - Given the tentative estimate of λ in the maximization
 - Given an initial value of the variance-covariance matrix, $\hat{\Sigma}_0$

VECH GARCH model

- Conditional variances and conditional covariance depend on all innovations and cross-products
- Two variable example with one lag of innovations squared and all lagged variance terms
 - Let a s and b s represent parameters

$$h_{11,t} = c_{11,0} + a_{11,1}u_{1,t-1}^2 + a_{12,1}u_{2,t-1}^2 + a_{13}u_{1,t-1}u_{2,t-1} + b_{11,1}h_{1,t-1}^2 + b_{12,1}h_{2,t-1}^2 + b_{13}h_{12,t-1}$$

$$h_{22,t} = c_{21,0} + a_{21,1}u_{1,t-1}^2 + a_{22,1}u_{2,t-1}^2 + a_{23}u_{1,t-1}u_{2,t-1} + b_{21,1}h_{1,t-1}^2 + b_{22,1}h_{2,t-1}^2 + b_{23}h_{12,t-1}$$

$$h_{12,t} = c_{31,0} + a_{31,1}u_{1,t-1}^2 + a_{32,1}u_{2,t-1}^2 + a_{33}u_{1,t-1}u_{2,t-1} + b_{31,1}h_{1,t-1}^2 + b_{32,1}h_{2,t-1}^2 + b_{33}h_{12,t-1}$$

- Everything depends on everything
- Problems with this model
 - Curse of dimensionality will overtake this quickly
 - No guarantee of positive variance-covariance matrix every period

Curse of Dimensionality with IGARCH

- Suppose a general IGARCH in which all variances and covariances depend on all lagged variances and cross-products of errors
- Two asset case

$$h_{11,t} = c_{11,0} + a_{11,1}u_{1,t-1}^2 + a_{12,1}u_{2,t-1}^2 + a_{13}u_{1,t-1}u_{2,t-1} + b_{11,1}h_{1,t-1}^2 + b_{12,1}h_{2,t-1}^2 + b_{13}h_{12,t-1}$$

$$h_{22,t} = c_{21,0} + a_{21,1}u_{1,t-1}^2 + a_{22,1}u_{2,t-1}^2 + a_{23}u_{1,t-1}u_{2,t-1} + b_{21,1}h_{1,t-1}^2 + b_{22,1}h_{2,t-1}^2 + b_{23}h_{12,t-1}$$

$$h_{12,t} = c_{31,0} + a_{31,1}u_{1,t-1}^2 + a_{32,1}u_{2,t-1}^2 + a_{33}u_{1,t-1}u_{2,t-1} + b_{31,1}h_{1,t-1}^2 + b_{32,1}h_{2,t-1}^2 + b_{33}h_{12,t-1}$$

- The number of parameters in each equation is $2m + 1$
- The number of equations is $m(m + 1) / 2$
- The total number of parameters is $m(m + 1)(2m + 1) / 2$
- Five assets implies $m(m + 1)(2m + 1) / 2 = 165$ parameters
- Ten assets implies $m(m + 1)(2m + 1) / 2 = 1105$ parameters

Diagonal VEC GARCH model

- The Diagonal VEC GARCH model has each term in the variance-covariance matrix evolve independently according to a GARCH(1,1)
- Two-variable example
 - Let a s and b s represent parameters

$$h_{11,t} = c_{11,0} + a_{11,1}u_{1,t-1}^2 + b_{11,1}h_{1,t-1}^2$$

$$h_{22,t} = c_{21,0} + a_{22,1}u_{2,t-1}^2 + b_{22,1}h_{2,t-1}^2$$

$$h_{12,t} = c_{31,0} + a_{33}u_{1,t-1}u_{2,t-1} + b_{33}h_{12,t-1}$$

- Still no guarantee that the estimated variance-covariance matrix will be positive definite in every period

Use of Cholesky decomposition in computations can guarantee positive definiteness

- We want the estimated variance-covariance matrix to be positive definite every period
 - Models such as the Diagonal VEC GARCH model with each term in the variance-covariance matrix evolving independently according to a GARCH(1,1) will not necessarily satisfy positive definiteness
- Transforming the estimation problem using a Cholesky decomposition ensures positive definiteness of the covariance matrix every period
 - Not sure it matters unless you are writing a program to estimate GARCH models or using a program that doesn't use this algorithm

Constant-correlation GARCH model

- Constant-correlation model

- A constant correlation is not an obvious constraint to impose with time-varying variances
 - Implies that the covariance varies proportionately to the standard deviations of the innovations
 - This implication can be seen from

$$\rho = \frac{\text{Cov}[u_{1t}, u_{2t}]}{\text{SD}[u_{1t}]\text{SD}[u_{2t}]} = \frac{h_{12,t}}{\sqrt{h_{11,t}}\sqrt{h_{22,t}}}$$

- Does allow for multivariate aspect of process
- A two-variable GARCH model is

$$\begin{bmatrix} \sigma_{11,t} \\ \sigma_{22,t} \end{bmatrix} = \begin{bmatrix} \alpha_{10} \\ \alpha_{20} \end{bmatrix} + \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} u_{1,t-1}^2 \\ u_{2,t-1}^2 \end{bmatrix} + \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} h_{11,t-1} \\ h_{22,t-1} \end{bmatrix}$$

- Do not need a term for covariance because its value each period is implied by the estimated constant correlation

- Two-variable example

- The constant-correlation model can be written

$$h_{11,t} = \alpha_{10} + \alpha_{11}u_{1,t-1}^2 + \alpha_{12}u_{2,t-1}^2 + \beta_{11}h_{11,t-1} + \beta_{12}h_{22,t-1}$$

$$h_{22,t} = \alpha_{20} + \alpha_{21}u_{1,t-1}^2 + \alpha_{22}u_{2,t-1}^2 + \beta_{21}h_{11,t-1} + \beta_{22}h_{22,t-1}$$

- Bollerslev determined the conditions necessary for this model to be stationary

Time-varying correlation GARCH

- Can have a time-varying correlation
 - One representation suggested based on a positive correlation is a GARCH(1,1) model for the variances combined with

$$\rho_{21,t} = \frac{\exp(q_t)}{1 + \exp(q_t)}$$
$$q_t = \bar{\omega}_0 + \bar{\omega}_1 \rho_{21,t-1} + \bar{\omega}_2 \frac{u_{1,t-1} u_{2,t-1}}{\sqrt{h_{11,t-1}} \sqrt{h_{22,t-1}}}$$

where $\bar{\omega}_0$, $\bar{\omega}_1$ and $\bar{\omega}_2$ are parameters to be estimated

- If the correlation can be negative, a better choice of function for the correlation would be

$$\rho_{21,t} = \frac{\exp(q_t) - 1}{\exp(q_t) + 1}$$