Financial Econometrics

Multivariate Time Series Analysis

Gerald P. Dwyer, Jr.

Trinity College, Dublin

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Vector autoregressions

- Simple autoregression

$$r_t = \phi_0 + \phi_1 r_t + a_t$$

First-order vector autoregression

- For a vector? First-order vector autoregression is

$$\mathbf{r}_t = \boldsymbol{\phi}_0 + \Phi_1 \mathbf{r}_{t-1} + \mathbf{a}_t$$

where there are k returns in the vector $\mathbf{r}_t$

$$\begin{aligned} \mathbf{r}_t' &= [r_{1t}, ..., r_{kt}] \\ \boldsymbol{\phi}_0' &= [\phi_{10}, ..., \phi_{k0}] \\ \Phi_1 &= \begin{bmatrix} \phi_{11} & \cdots & \phi_{1k} \\ & \cdots & \\ \phi_{k1} & & \phi_{kk} \end{bmatrix}, \\ \mathbf{a}_t' &= [a_{1t}, ..., a_{kt}] \end{aligned}$$

where for ϕ_{ij}, i is equation, j is variable in equation

- Common to put vectors and matrices in bold and I'll do that generally
  - Can't do that for Φ_1 here because bold version is not rendered correctly

Estimation Strategy

- Common practice is to use Ordinary Least Squares (OLS) to estimate this set of equations
  - Why?
  - Same variables in every equation – Therefore OLS is equivalent to Seemingly Unrelated Regression

Bayesian Vector Autoregression

- A Bayesian Vector Autoregression commonly has a “Minnesota prior”: The prior supposes that each series is a random walk
    - Data then update this prior
    - Improves precision of estimated coefficients and, to some degree, starts with a simple characterization that is consistent with the data

Reduced form and structural equations

- Can interpret a VAR as a reduced form of some equation system
  - Two-variable VAR

$$
\begin{aligned}
m_t &= \phi_{10} + \phi_{11} m_{t-1} + \phi_{12} p_{t-1} + a_{1t} \\
p_t &= \phi_{20} + \phi_{21} m_{t-1} + \phi_{22} p_{t-1} + a_{2t}
\end{aligned} \quad (1)
$$

  - This has no simultaneous determination
- Suppose economic theory suggests

$$
\begin{aligned}
m_t &= \alpha_m + \beta_{1,11} m_{t-1} + \beta_{1,20} p_t + \beta_{1,21} p_{t-1} + \varepsilon_t^m \\
p_t &= \alpha_p + \beta_{2,10} m_t + \beta_{2,11} m_{t-1} + \beta_{2,21} p_{t-1} + \varepsilon_t^p
\end{aligned}
$$

  - $\{\varepsilon_t^m\}$ and $\{\varepsilon_t^p\}$ are zero mean, constant variance, serially uncorrelated processes
  - $\beta_{1,20}$ may reflect effect of prices on the nominal quantity of money demanded
  - $\beta_{2,10}$ may reflect effect of supply of money on prices
- Can solve for reduced form and it will look exactly like the VAR reduced form (1)

Identification of structural equations

- Suppose have simultaneous system

$$
\begin{aligned}
m_t &= \alpha_m + \beta_{1,11} m_{t-1} + \beta_{1,20} p_t + \beta_{1,21} p_{t-1} + \varepsilon_t^m \\
p_t &= \alpha_p + \beta_{2,10} m_t + \beta_{2,11} m_{t-1} + \beta_{2,21} p_{t-1} + \varepsilon_t^p
\end{aligned}
$$

  - The coefficients $\beta_{1,20}$ and $\beta_{2,10}$ are not identified
    - "Not identified" means "cannot be estimated separately using a time-series dataset"
    - Could be identified with the right additional variables or restrictions on the coefficients

- Suppose solve for reduced form

$$
\begin{aligned}
m_t &= \phi_{10} + \phi_{11} m_{t-1} + \phi_{12} p_{t-1} + \varepsilon_t^m \\
p_t &= \phi_{20} + \phi_{21} m_{t-1} + \phi_{22} p_{t-1} + \varepsilon_t^p \qquad (2)
\end{aligned}
$$

- It is common to proceed to "identify" the contemporaneous effect of m on p or p on m by including it in one or the other regression

Recursive version

- Inclusion of contemporary variable in one of two equations (**recursive system**)

$$
\begin{aligned}
m_t &= b_{10} + b_{11} m_{t-1} + b^* p_t + b_{12} p_{t-1} + e_t^m \\
p_t &= \phi_{20} + \phi_{21} m_{t-1} + \phi_{22} p_{t-1} + e_t^p
\end{aligned}
$$

  - Estimation by OLS implies that the errors in the two equations are uncorrelated
  - Can compute from OLS estimation of VAR using Cholesky decomposition
- How do these coefficients relate to **simultaneous system**

$$
\begin{aligned}
m_t &= \alpha_m + \beta_{1,11} m_{t-1} + \beta_{1,20} p_t + \beta_{1,21} p_{t-1} + \varepsilon_t^m \\
p_t &= \alpha_p + \beta_{2,10} m_t + \beta_{2,11} m_{t-1} + \beta_{2,21} p_{t-1} + \varepsilon_t^p
\end{aligned}
$$

or to reduced form **VAR**

$$
\begin{aligned}
m_t &= \phi_{10} + \phi_{11} m_{t-1} + \phi_{12} p_{t-1} + \varepsilon_t^m \\
p_t &= \phi_{20} + \phi_{21} m_{t-1} + \phi_{22} p_{t-1} + \varepsilon_t^p
\end{aligned}
$$

Relationship between recursive system, simultaneous system and VAR

- Clearly recursive system, simultaneous system and VAR do not represent the same things
- VAR is a reduced form of simultaneous system
  - Coefficients in VAR are functions of the coefficients in the simultaneous system
  - Error terms in VAR are a combination of the error terms in the simultaneous system
- Recursive system is a version of the VAR in which all of the correlation between ε_t^m and ε_t^p in the VAR is reflected in the coefficient of p_t in the m_t equation in the recursive system
- There is no necessary reason that the coefficients in the simultaneous system should equal the coefficients in the recursive system
  - The coefficients will be equal if the behavioral simultaneous system is recursive
    - The coefficient of m_t in the p_t equation is zero
    - And the errors are uncorrelated

Higher-order autoregressions

- Can write a *p*-th order autoregression as

$$\mathbf{r}_t = \boldsymbol{\phi}_o + \sum_{i=1}^{p} \Phi_i \mathbf{r}_{t-i} + \mathbf{a}_t$$

- Lag operator can be handy, $\mathrm{L}\,\mathbf{r}_t = \mathbf{r}_{t-1}$ and $\mathrm{L}^i\,\mathbf{r}_t = \mathbf{r}_{t-i}$
- This implies $\Phi_i \mathbf{r}_{t-i} = \Phi_i\,\mathrm{L}^i\,\mathbf{r}_t$ and

$$\sum_{i=1}^{p} \Phi_i \mathbf{r}_{t-i} = \sum_{i=1}^{p} \Phi_i\,\mathrm{L}^{i-1}\,\mathbf{r}_t = \Phi\left(L\right)\mathbf{r}_{t-1}$$

  so

$$\mathbf{r}_t = \boldsymbol{\phi}_o + \Phi\left(L\right)\mathbf{r}_{t-1} + \mathbf{a}_t$$

- Higher-order autoregression doesn't add anything of substance, at least formally
  - Does add a lot of parameters

Estimation of higher-order autoregressions

- A fairly common practice in economics is to use the same lag length for all equations for all variables
  - Contributes to Zellner calling them "Very Awful Regressions"
  - How decide reliably that some coefficients should be zero?
  - Suppose that lags one and four have t-ratios of 3 and lags two and three have t-ratios around one
    - Are the coefficients really zero or just imprecisely estimated?
  - Many extraneous coefficients leads to wide confidence intervals for forecasts
- Determine lag length same way as for univariate autoregressions
  - F-ratios until five percent "significance"
  - Akaike information criterion
  - (Schwarz) Bayesian information criterion

Cointegration and vector error correction mechanism

- Two variables are cointegrated if a linear combination of the variables has fewer unit roots than the original set of variables
- Empirical strategy
  - Test for unit roots in original series
    - Augmented Dickey-Fuller
    - Phillips-Perron
  - Test for unit roots in set of variables
    - Johansen test is the best generally speaking
    - If you know the value of the coefficient that should relate two variables, e.g. asset prices by coefficient of one, then an augmented Dickey-Fuller unit root test on that linear combination is better
    - Suppose x_{1t} and x_{2t} each have one unit root
    - Suppose $z_t = x_{1t} - x_{2t}$ has no unit root, but estimate $\widehat{z}_t = x_{1t} - \widehat{\beta} x_{2t}$, $\widehat{\beta} \neq 1$ in general
    - Then $\widehat{z}_t = x_{1t} - \widehat{\beta} x_{2t-1} = z_t + \left(1 - \widehat{\beta}\right) x_{2t}$, and you might infer that z_t has a unit root because $\widehat{z}_t$ does, even though z_t does not have a unit root (Why does $\widehat{z}_t$ have a unit root?)

Cointegration and Moving Average Representation

- If variable have unit roots but are cointegrated, then there exists a Vector Error Correction Mechanism
  - Two-variable system, r_{1t} and r_{2t}
  - Let $r_{it} = p_{it} - p_{it-1}$
  - Suppose that p_{1t} and p_{2t} are cointegrated with $p_{it} - p_{it}$ not having a unit root
  - Cointegrating vector is $\beta' = \begin{bmatrix} 1 & -1 \end{bmatrix}$
  - Therefore

$$\beta' \mathbf{p}_{t-1} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} p_{1t} \\ p_{2t} \end{bmatrix}$$

  - By a generalization of Wold's Theorem to multivariate processes, there exists a vector moving average representation

$$\mathbf{r}_t = \mathbf{w}_0 + \sum_{i=1}^{\infty} \mathbf{w}_i \mathbf{a}_{t-i}$$

Cointegration, VARs and ECM

- The generalization of Wold's Theorem

$$\mathbf{r}_t = \mathbf{w}_0 + \sum_{i=1}^{\infty} \mathbf{w}_i \mathbf{a}_{t-i}$$

  indicates that there is a covariance stationary representation of multivariate series
- This **cannot** be rewritten as an autoregressive representation

$$\mathbf{r}_t = \boldsymbol{\phi}_0 + \sum_{i=1}^{\infty} \Phi_i \mathbf{r}_{t-i} + \mathbf{a}_t$$

- **There does exist an Error Correction Representation**

$$\mathbf{r}_t = \boldsymbol{\phi}_0 + \boldsymbol{\alpha}\boldsymbol{\beta}' \mathbf{p}_{t-1} + \sum_{i=1}^{\infty} \Phi_i \mathbf{r}_{t-i} + \mathbf{a}_t$$

  - This generally is called an "Error Correction Mechanism" or "ECM" for short

ECM and Long-run relationship

- This Error Correction Mechanism (ECM)

$$\mathbf{r}_t = \boldsymbol{\phi}_0 + \boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{p}_{t-1} + \sum_{i=1}^{\infty} \Phi_i \mathbf{r}_{t-i} + \mathbf{a}_t$$

has interesting steady-state behavior

-
    - The term $\boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{p}_{t-1}$, which does not have unit root by hypothesis, forces the return of p_{1t} and p_{2t} to the cointegrated relationship with p_{1t} equal to p_{2t}

Long run relationship

- ECM

$$\mathbf{r}_t = \boldsymbol{\phi}_0 + \boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{p}_{t-1} + \sum_{i=1}^{\infty} \Phi_i \mathbf{r}_{t-i} + \mathbf{a}_t$$

- Steady-state behavior can be inferred by setting the innovations $\mathbf{a}_t$ to zero and supposing that $\mathbf{r}_t$ and $\mathbf{p}_t$ are constant
  - If $\mathbf{p}$ is constant, then $\mathbf{r}$ is zero
  - This implies $\boldsymbol{\phi}_0 + \boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{p} = \mathbf{0}$
  - In the first equation, $\phi_{01} + \alpha_1 (p_2 - p_1) = 0$
  - And therefore $p_2 - p_1 = -\phi_{01}/\alpha_1$
  - If expect $p_2 - p_1 = 0$, then also expect $\phi_{01} = 0$
  - Therefore, this equation does imply $p_2 = p_1$ if $\boldsymbol{\phi}_0 = \mathbf{0}$

Characterization of Long-run relationship

- In the ECM

$$\mathbf{r}_t = \boldsymbol{\phi}_0 + \boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{p}_{t-1} + \sum_{i=1}^{\infty} \Phi_i \mathbf{r}_{t-i} + \mathbf{a}_t$$

- The variable $\boldsymbol{\beta}'\mathbf{p}_{t-1}$, which does not have unit root by hypothesis, makes sure that the dynamics of p_{1t} and p_{2t} are consistent with the long-run relationship
    - Some people would call $p_1 - p_2 = 0$ in this ECM an "equilibrium" relationship
    - In a statistical sense, that is reasonable
    - Without an economic model, this is not a particularly good use of the word "equilibrium"
    - A more careful use of English would be the "stationary" or "steady state" relationship

ECM and Speed of Adjustment

- The coefficient vector $\boldsymbol{\alpha}$ often is called the "speed of adjustment" in the ECM

$$\mathbf{r}_t = \boldsymbol{\phi}_0 + \boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{p}_{t-1} + \sum_{i=1}^{\infty} \Phi_i \mathbf{r}_{t-i} + \mathbf{a}_t$$

  - The coefficient vector $\boldsymbol{\alpha}$ is the partial derivative of $\mathbf{r}_t$ with respect to deviations from the steady state relationship $\boldsymbol{\beta}'\mathbf{p}_{t-1}$
  - This does not show the response of $\mathbf{r}_t$ to $\boldsymbol{\beta}'\mathbf{p}_{t-1}$ over time to a shock to $\mathbf{a}_t$

ECM and Impulse Response

- The impulse response in the ECM
$$\mathbf{r}_t = \boldsymbol{\phi}_0 + \boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{p}_{t-1} + \sum_{i=1}^{\infty} \Phi_i \mathbf{r}_{t-i} + \mathbf{a}_t$$
is the response to a "shock", an "innovation" in the error term $\mathbf{a}_t$
- Computing the impulse response function requires iterating the equation
  - Start from stead state consistent with theory
    - Suppose that $\boldsymbol{\phi}_0 = \mathbf{0}$
    - Suppose that all variables are in the steady state, which could involve $\mathbf{r}_t$ equal to expected returns but we simplify with $\mathbf{r}_t = \mathbf{0}$
  - Suppose that there is a unit shock to a_{1t} at $t = 0$, so $a_{10} = 1$
  - Then $r_{10} = 1 = p_{10} - p_{1,-1}$ and suppose $p_{1,-1} = 0$ for simplicity, so $p_{10} = 1$
  - Then $p_{10} - p_{20} = 1$ and $\mathbf{r}_1$ will respond to $\boldsymbol{\beta}'\mathbf{p}_{t-1} \neq \mathbf{0}$
  - But $r_{10} \neq 0$ also because $r_{10} = 1$
  - Therefore the response of $\mathbf{r}_1$ to the shock to a_{10} will depend on both $\boldsymbol{\alpha}$ and Φ_1

Computing the Impulse Response Function

- The impulse response in the ECM

$$\mathbf{r}_t = \boldsymbol{\phi}_0 + \boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{p}_{t-1} + \sum_{i=1}^{\infty} \Phi_i \mathbf{r}_{t-i} + \mathbf{a}_t$$

  is computed by iterating the entire equation
- The impulse response depends on all of the coefficients
- It's not hard or particularly complicated to do the underlying algebra

Impulse response functions of adjusted basis for S&P 500 to a futures shock